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Ergodic theorems for random clusters

Michael Björklund*

Department of Mathematics, Royal Institute of Technology, Lindstedtsvägen 25, S-100 44 Stockholm, Sweden

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Abstract

We prove pointwise ergodic theorems for a class of random measures which occurs in Laplacian growth models, most notably in the anisotropic Hastings–Levitov random cluster models. The proofs are based on the theory of quasi-orthogonal functions and uniform Wiener–Wintner theorems.

Keywords: Ergodic theorems; Loewner evolution; Weak convergence

1. Introduction

1.1. General comments

This paper is concerned with the asymptotic behavior of a class of random measures which arise in Laplacian growth models. The main result is an ergodic theorem for these measures, which has been used by Johansson, Sola and Turner in [3] to establish the existence of a limit cluster in anisotropic Hastings–Levitov models. We refer to their paper for further details and motivation.

1.2. Statement of theorems

Let $(X, \mathcal{F}, \mu)$ be a standard probability measure space, and suppose that g_k is a sequence of bounded and independent, identically distributed real-valued measurable functions on $(X, \mathcal{F})$. Let Z be a compact and metrizable space, and suppose that θ_k is an ergodic Markov chain on Z, defined on $(X, \mathcal{F}, \mu)$, with stationary measure λ and independent of the family g_k. We assume

* Tel.: +46 736316097.
E-mail address: mickebj@math.kth.se.

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that the Markov chain θ_k has a spectral gap with respect to separable and dense normed subspace $\mathcal{H}$ of the space of real-valued continuous functions f on Z. This means that for every k, we have $(\theta_k)_*\mu = \lambda$ and there are constants $C > 0$ and $0 \le \tau < 1$, such that for all f in $\mathcal{H}$ and $k, l \ge 1$, we have the inequality,

$$\left| \int_X f(\theta_k(x))\, f(\theta_l(x))\, \mathrm{d}\mu(x) - \left(\int_Z f\, \mathrm{d}\lambda \right)^2 \right| \le C \| f \|_{\mathcal{H}}^2\, \tau^{|k-l|},$$

where $\| \cdot \|_{\mathcal{H}}$ denotes the norm in $\mathcal{H}$. We say that the Markov chain θ_k is *contractive on* $\mathcal{H}$ if the above inequality holds. Note that this is always the case when all θ_k are independent and identically distributed.

Suppose that M_n is a martingale defined with respect to some filtration on $(X, \mathcal{F})$, and assume that for some $p > 1$, we have

$$\sum_{n \ge 1} \frac{\| M_n \|_p^p}{n^p} < \infty.$$

In particular, this is true for $p \ge 4$ if M_n is the sum of a sequence of independent and identically distributed random variables in $L^4(X, \mu)$ with mean zero.

Let $t_{k,n} = k/n$, and define the sequences $\widetilde{M}_{k,n} = M_k/n + t_{k,n}$ and

$$Y_{k,n} = \left(\theta_k, \widetilde{M}_{k,n}\right) \quad \text{on } (X, \mathcal{F}).$$

Suppose that (ρ_k) is a quasi-orthogonal sequence (see Definition 3.1) in $L^2(X)$, and $\beta \in \mathbb{R}$. Define the following measurable array,

$$\widetilde{\rho}_{k,n}(x) = \frac{1}{n}(\beta + \rho_k(x)), \quad 1 \le k \le n.$$

We can now formulate the main theorem in this paper.

Theorem 1. *There is a conull subset X' of X such that for all x in X',*

$$\lim_{n \to \infty} \sum_{k=1}^{n} \widetilde{\rho}_{k,n}(x)\, \delta_{\widetilde{M}_{k,n}(x)} = \beta\, m_{[0,1]},$$

in the weak topology on $\mathcal{M}(\mathbb{R})$, where $m_{[0,1]}$ denotes the normalized Lebesgue measure on the interval $[0, 1]$.

Before we embark on the proof of Theorem 1, we discuss an application of the theorem to contractive Markov chains. Suppose that there is a separable and dense subspace of $\mathcal{H}$ on which the Markov chain θ_k above is contractive with an ergodic and stationary measure λ, and let g_k be a sequence of non-negative bounded independent and identically distributed random variables on X. Then we have the following corollary of Theorem 1.

Corollary 1.1. *There is a conull subset X' of X such that for all x in X', we have the limit*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} g_k(x)\, \delta_{Y_{k,n}(x)} = \left(\int_X g_1\, \mathrm{d}\mu \right) \lambda \otimes m_{[0,1]}$$

in the weak topology on $\mathcal{M}(Z \times \mathbb{R})$, where $m_{[0,1]}$ denotes the normalized Lebesgue measure on $[0, 1]$.

Corollary 1.1 admits a generalization to general ergodic weights (g_k). Let $(X, \mathcal{B}_X, \mu_X, T)$ be an ergodic probability measure preserving system. Then we have the following theorem.

Theorem 2. *Suppose that $f \in L_0^\infty(X, \mu)$, and $\beta \in \mathbb{R}$. Then there is a conull set $X' \subseteq X$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (\beta + f(T^k x))\, \delta_{\widetilde{M}_{k,n}(x)} = \beta\, m_{[0,1]},$$

in the weak topology on $\mathcal{M}(\mathbb{R})$.

The paper is organized as follows. In Section 2 we describe the Hastings–Levitov models and give motivation for Corollary 1.1. In Sections 3 and 4 we give complete proofs of Theorems 1 and 2 respectively.

2. Hastings–Levitov models

In this section we describe the main application of Corollary 1.1. More details can be found in the paper [3]. Let Δ denote the exterior disk

$$\Delta = \{z \in \mathbb{C}_\infty \mid |z| > 1\},$$

where $\mathbb{C}_\infty$ is the Riemann sphere, and consider the unique conformal mapping

$$\phi_\delta : \Delta \to \Delta \setminus (1, 1+\delta],$$

with $\phi_\delta(z) = C(\delta)z + O(1)$, $C(\delta) > 0$ at infinity. Let $\theta_1, \theta_2, \ldots$ be i.i.d. random variables on the unit circle $\mathbb{T}$ with common law ν, and let $\delta_1, \delta_2, \ldots$ be positive independent random variables with laws $\sigma_1, \sigma_2, \ldots$, and independent of $\theta_1, \theta_2, \ldots$. Define

$$\phi_{\delta_n}^{\theta_n}(z) = \mathrm{e}^{\mathrm{i}\theta_n} \phi_{\delta_n}(\mathrm{e}^{-\mathrm{i}\theta_n} z).$$

We set $\Phi_0(z) = z$ and

$$\Phi_n(z) = \Phi_n \circ \phi_{\delta_n}^{\theta_n}(z).$$

This will produce a sequence of random conformal maps $\Phi_n : \Delta \to \mathbb{C} \setminus K_n$ with K_n compact and $K_{n-1} \subset K_n$. We will refer to the sets K_n as *random clusters*.

Recall that a decreasing Loewner chain is a family of conformal mappings

$$f_t : \Delta \to \mathbb{C} \setminus K_t,$$

with $f_t(\infty) = \infty$ and $f_t'(\infty) > 0$ and

$$K_{t_1} \subset K_{t_2} \quad \text{if } t_1 < t_2.$$

It is well known that decreasing Loewner chains can be parameterized by Schwarz–Herglotz integrals of probability measures on $\mathbb{T}$. More precisely, every decreasing Loewner chain will satisfy the *Loewner–Kufarev equation*

$$\partial_t f_t(z) = z f_t'(z) \int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta}\, \mathrm{d}\mu_t(\zeta),$$

with initial condition $f_0(z) = z$, and where μ_t is a sequence of probability measures on $\mathbb{T}$ with certain properties.

We will think of the family $\{\mu_t\}_{t\geq 0}$ as a locally bounded measure on $\mathbb{R} \times \mathbb{T}$ via the correspondence,

$$\mu(\varphi) = \int_{\mathbb{R}} \left(\int_{\mathbb{T}} \varphi(\zeta, t) \, \mathrm{d}\mu_t(\zeta) \right) \mathrm{d}t,$$

where φ is a compactly supported continuous function on $\mathbb{R} \times \mathbb{T}$.

In this paper we will establish almost sure convergence of sequences μ^N of random locally bounded measures on $\mathbb{T} \times \mathbb{R}$, defined on some common probability measure space, of the form $\mathrm{d}\mu^N(\zeta, t) = \mathrm{d}\mu_t^N(\zeta)\chi_{[0,1]}(t)\,\mathrm{d}t$, with

$$\int_{\mathbb{T}} f(\zeta) \, \mathrm{d}\mu_t^N(\zeta) = \frac{1}{N} \sum_{k=1}^{N} X_k f(\theta_k) \delta_{T_{k,N}}(t),$$

where f is a continuous function on $\mathbb{T}$ and

$$T_{k,N} = \frac{1}{N} \sum_{j=1}^{k} X_j,$$

where $X_1, X_2, \ldots$ are positive i.i.d. random variables, independent of $\theta_1, \theta_2, \ldots$. The motivation for these choices can be found in [3]. It turns out that μ_t^N corresponds to the law of the conformal mapping Φ_N where the slit lengths δ_j^N, $j = 1, \ldots, N$ are distributed according to X_j/N. We are interested in the asymptotic behavior of these measures. Note that if the X_j's are bounded, the sequence μ^N is almost surely tight and bounded. Corollary 1.1 was used by Johansson, Sola and Turner [3] to establish the existence of a limit cluster for the Loewner chain constructed above with $\mu_t = \nu$.

3. Proof of the main theorem

3.1. Random measures

Let E be a proper metric space, i.e. closed and bounded subsets are compact. Let $\mathcal{M}(E)$ denote the real vector space of signed measures on E. A sequence η_n in $\mathcal{M}(E)$ is said to converge *vaguely* to a measure η in $\mathcal{M}(E)$ if

$$\lim_{n\to\infty} \int_E f \, \mathrm{d}\eta_n = \int_E f \, \mathrm{d}\eta.$$

for all compactly supported continuous functions on E. The sequence converges *weakly* if the same is true for bounded and continuous functions. A subset $\mathcal{A}$ of $\mathcal{M}(E)$ is called *tight* if for all $\varepsilon > 0$, there is a compact set K such that $|\nu|(K^c) < \varepsilon$ for all ν in $\mathcal{A}$, where $|\cdot|$ denotes the total variation. We say that $\mathcal{A}$ is bounded if there is a constant C such that $|\nu|(E) < C$ for all ν in $\mathcal{A}$. We will use the following theorem by Baez–Duarte [2].

Theorem 3. *A sequence of signed measures η_n converges weakly to η if η_n converges vaguely to η and the sequence is bounded and tight.*

Since the space $C_c(E)$ of continuous functions with compact support in E is separable, it suffices to establish the vague limit above only for a countable number of elements in $C_c(E)$. If, in addition, the sequence is tight and bounded, the same is true for weak convergence, in view of Theorem 3.

Let $(X, \mathcal{F}, \mu)$ be a probability space. A *random measure* on E is a measurable map π from X into $\mathcal{M}(E)$, equipped with the Borel structure coming from the weak topology. We will consider sequences of random measures, defined on some common probability space $(X, \mathcal{F}, \mu)$, with the property that there is a conull subset of X' such that the sequence π_n^x of $\mathcal{M}(E)$ is tight and bounded for all $x \in X'$ (but not necessarily uniformly, i.e. we do not require that the compact set K above can be chosen independently of x, nor that the sequence is almost everywhere uniformly bounded on X). Thus, if we are able to construct conull subsets X_f of X for all f in a countable dense subset of $C_c(E)$, with the property that there is a random measure π on E, independent of f, such that

$$\lim_{n\to\infty} \int_E f \, \mathrm{d}\pi_n^x = \int_E f \, \mathrm{d}\pi^x,$$

for all $x \in X_f$, then by taking $X'' = X' \cap \left(\cap_f X_f\right)$, which is again conull, we conclude that the measures π_n^x converges vaguely (and thus weakly, since the sequence π_n^x is assumed to be almost surely tight and bounded) to π^x for all x in the set X''.

3.2. Fourier analysis

Let $\mathcal{S}$ denote the Schwarz class on $\mathbb{R}$ (see e.g. [4]). For ϕ in $\mathcal{S}$ we define the Fourier transform to be the function,

$$\hat{\phi}(\xi) = \int_{\mathbb{R}} \phi(x)\, \mathrm{e}^{-\mathrm{i}\xi x}\, \mathrm{d}x, \quad \xi \in \mathbb{R}.$$

It is a standard fact that $\hat{\phi}$ is in $\mathcal{S}$ and that ϕ can be reconstructed from $\hat{\phi}$ by the formula,

$$\phi(x) = \int_{\mathbb{R}} \hat{\phi}(\xi)\, \mathrm{e}^{\mathrm{i}\xi x}\, \mathrm{d}m(\xi),$$

where m denotes the Plancherel measure on $\mathbb{R}$, see [4].

3.3. Quasi-orthogonal functions

We recall the definition of a quasi-orthogonal sequence in a Hilbert space.

Definition 3.1. Let $(X, \mathcal{F}, \mu)$ be a measure space. A sequence ρ_k in $L^2(X, \mu)$ is called *quasi-orthogonal* if there is a constant $C > 0$, such that for all c in $\ell^2(\mathbb{N})$, we have the inequality,

$$\left| \sum_{k,l} c_k \overline{c_l} \langle \rho_k, \rho_l \rangle \right| \le C \sum_k |c_k|^2.$$

In particular, a sequence ρ_k in $L^2(X, \mu)$ is quasi-orthogonal if there are constants C and $0 \le \tau < 1$ such that

$$|\langle \rho_k, \rho_l \rangle| \le C \tau^{|k-l|}, \quad \forall\, k, l \ge 1.$$

The following theorem by Kac, Salem and Zygmund [5] shows that the convergence theory of quasi-orthogonal sequences is not too different to the theory of orthogonal sequences.

Theorem 4. *Suppose that ρ_k is a quasi-orthogonal sequence in $L^2(X, \mu)$. There is a constant C such that for every triangular array of complex numbers a_{nk}, $k = 1, \ldots, n$, and $n \ge 1$, we have* *the inequality,*

$$\int_X \sup_{1\le m\le n} \left| \sum_{k=1}^{m} a_{mk}\, \rho_k(x) \right|^2 \mathrm{d}\mu(x) \le C \sum_{k=1}^{n} |a_{nk}|^2 \|\rho_k\|_2^2 \log^2(1+k).$$

In particular, if a_{nk} is a triangular array with the property that

$$\lim_{n\to\infty} \sum_{k=1}^{n} |a_{nk}|^2\, \|\rho_k\|_2^2 \log^2(1+k) = 0,$$

then we have the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk}\, \rho_k(x) = 0$$

almost everywhere on X. Similar results have been attained by Teicher [6] for independent random variables.

3.4. Main ergodic theorem

We first indicate how Corollary 1.1 follows from Theorem 1. Recall the notation introduced in Section 1.2. Let Z be a compact and metrizable space, and suppose that $\mathcal{H}$ is a separable and dense normed subspace of $C(Z)$. Let θ_k be a contractive Markov chain on $\mathcal{H}$ with an ergodic and stationary measure λ. For $n \ge 1$, we define,

$$Y_{k,n} = (\theta_k, \widetilde{M}_{k,n}), \quad 1 \le k \le n.$$

Finally, we assume that g_k is a sequence of bounded and identically distributed non-negative random variables on X, which are independent of the Markov chain θ_k.

Corollary 3.1. *There is a conull subset X' of X such that for all x in X',*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g_k(x)\, \delta_{Y_{k,n}(x)} = \left(\int_X g_1\, \mathrm{d}\mu \right) \lambda \otimes m_{[0,1]}$$

in the weak topology on $\mathcal{M}(Z \times \mathbb{R})$.

Proof of Corollary 3.1. We can without loss of generality assume that $\int_X g_1\, \mathrm{d}\mu = 1$. Since the sequence of signed measures on $Z \times \mathbb{R}$,

$$\nu_n = \frac{1}{n} \sum_{k=1}^{n} g_k(x)\, \delta_{Y_{k,n}(x)}, \quad n \ge 1,$$

is bounded and tight, it suffices to prove the convergence of $\nu_n(f \otimes \phi)$ to $\lambda(f)\, m_{[0,1]}(\phi)$ for f and ϕ in dense and countable collections of functions in $\mathcal{H}$ and $C_o(\mathbb{R})$ respectively. Since g_k and θ_k are independent, the sequence

$$\widetilde{\rho}_{k,n} = g_k f(\theta_k)/n, \quad 1 \le k \le n,$$

is again quasi-orthogonal on $(X, \mathcal{F}, \mu)$ with $\int_X g_k\, f(\theta_k)\, \mathrm{d}\mu = \lambda(f)$. By Theorem 1, there exists, for every fixed f in $\mathcal{H}$, a conull subset X_f of X such that for all x in X_f,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g_k(x)\, f(\theta_k(x))\, \delta_{\widetilde{M}_{k,n}(x)} = \left(\int_Z f\, \mathrm{d}\lambda \right) m_{[0,1]},$$

in the weak topology on $\mathcal{M}(\mathbb{R})$. Since $\mathcal{H}$ is countable and dense in $C(Z)$, the intersection X' of all X_f with f in $\mathcal{H}$ is still conull. $\square$

In order to establish Theorem 1 we need some lemmata. The first lemma reduces the analysis to Riemannian integration theory.

Lemma 1. *There is a conull subset X' of X such that for all x in X',*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \widetilde{\rho}_{k,n}(x)\big(\delta_{\widetilde{M}_{k,n}(x)} - \delta_{t_{k,n}}\big) = 0,$$

in the weak topology on $\mathcal{M}(\mathbb{R})$.

Proof. Since the sequence of measures is clearly bounded and tight, it suffices to prove, for every fixed choice of ϕ in a countable and dense collection $\mathcal{C}$ of functions in $C_o(\mathbb{R})$, the existence of a conull subset X_ϕ of X with the property that for all x in X_ϕ, we have the limit,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \widetilde{\rho}_{k,n}(x)\big(\phi(\widetilde{M}_{k,n}(x)) - \phi(t_{k,n})\big) = 0.$$

We define X' as the intersection of all X_ϕ when ϕ ranges over all elements in $\mathcal{C}$.

We can choose the collection $\mathcal{C}$ to be contained in $\mathcal{S}$ so that Fourier transform techniques become available. For a fixed ϕ in $\mathcal{C}$, we pick $\varepsilon > 0$ and choose $r > 0$ such that

$$\int_{|\xi|>r} |\hat{\phi}(\xi)|\,\mathrm{d}m(\xi) < \frac{\varepsilon}{4}.$$

Define, for $|\xi| < r$, the function

$$R_n(x) = \int_{\mathbb{R}} \left[\sum_{k=1}^{n} \widetilde{\rho}_{k,n}(x)\mathrm{e}^{\mathrm{i}\xi t_{k,n}} \left(\mathrm{e}^{\mathrm{i}\xi M_{k,n}(x)} - 1\right)\right] \hat{\phi}(\xi)\,\mathrm{d}m(\xi).$$

Choose $\delta > 0$ such that

$$|\mathrm{e}^{\mathrm{i}\xi t} - 1| < \inf_{k,n\geq 1} \frac{\varepsilon}{4\|\hat{\phi}\|_\infty \|\widetilde{\rho}_{k,n}\|_\infty}$$

for all $|t| < \delta$ and $|\xi| < r$. This is possible since $\widetilde{\rho}_{k,n}$ is uniformly bounded from above in k and n. Define the maximal function

$$L_n = \sup_{1\leq k\leq n} |M_k|, \quad n \geq 1,$$

and the sets

$$A_n(\delta) = \{x \in X \mid L_n(x) < \delta n\}, \quad n \geq 1.$$

Note that if x is in $A_n(\delta)$, then $|R_n(x)| \leq \varepsilon$. Thus, if

$$B_n(\varepsilon) = \{x \in X \mid |R_n(x)| > \varepsilon\},$$

we have $A_n(\delta) \cap B_n(\varepsilon) = \emptyset$ for all $n \geq 1$. By the Borel–Cantelli Lemma and Doob's Maximal

Theorem,

$$\begin{aligned}\sum_{n\geq 1}\mu(B_n(\varepsilon)) &= \sum_{n\geq 1}\mu(B_n(\varepsilon)\cap A_n(\delta)) + \sum_{n\geq 1}\mu(B_n(\varepsilon)\cap A_n(\delta)^c)\\ &\leq \sum_{n\geq 1}\mu(A_n(\delta)^c) \leq \frac{1}{\delta^p}\sum_{n\geq 1}\frac{\|L_n\|_p^p}{n^p}\\ &\leq \frac{C^p}{\delta^p}\sum_{n\geq 1}\frac{\|M_n\|_p^p}{n^p} < +\infty,\end{aligned}$$

for all $\varepsilon > 0$, and hence $R_n(x) \to 0$ almost everywhere on X. □

The existence of the limit of the remaining sequence of random measures can be established using quasi-orthogonal functions as the following lemma show.

Lemma 2. *There is a conull subset X' of X such that for all x in X',*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\widetilde{\rho}_{k,n}\,\delta_{t_{k,n}} = \beta\, m_{[0,1]}$$

in the weak topology on $\mathcal{M}(\mathbb{R})$.

Proof. Let $\beta_{k,n} = \beta/n$. By the theory of Riemann integrals, we have

$$\sum_{k=1}^{n}\beta_{k,n}\,\delta_{t_{k,n}} = \beta\, m_{[0,1]},$$

thus it suffices to prove that there is a conull subset X' of X such that for all x in X', we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\rho_{k,n}\,\delta_{t_{k,n}} = 0,$$

in the weak topology on $\mathcal{M}(\mathbb{R})$. Since the sequence above is bounded and tight, we only need to establish the limit for a countable and dense collection $\mathcal{C}$ of functions in $C_o(\mathbb{R})$. By the countable additivity of measures, it suffices to prove that, for every fixed ϕ in $\mathcal{C}$, there exists of a conull subset X_ϕ such that for all x in X_ϕ,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\rho_{k,n}(x)\,\phi(t_{k,n}) = 0.$$

We define X' as the intersection of all X_ϕ for ϕ in $\mathcal{C}$, which again is a conull subset of X. By Theorem 4, with $n\, a_{nk} = \phi(t_{k,n})$, we have the maximal inequality,

$$\int_X \sup_{1\leq m\leq n}\left|\sum_{k=1}^{m}a_{mk}\rho_k(x)\right|^2 \mathrm{d}\mu(x) \leq C\sum_{k=1}^{n}a_{nk}^2\log^2(1+k)\|\rho_k\|_2^2.$$

Since $\|\rho_k\|_2$ is bounded in k, we conclude, by an elementary Borel–Cantelli argument, that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\rho_{k,n}\,\phi(t_{k,n}) = 0,$$

almost everywhere on X. □

4. Generalization to dependent weights

We now sketch a generalization of Corollary 1.1 to the case of not necessarily independent weights g_k. Let $(X, \mathcal{B}_X, \mu_X, T)$ be an ergodic probability measure preserving system. Recall that the *Kronecker factor* of $(X, \mathcal{B}_X, \mu_X, T)$ is the smallest sub-σ-algebra $\mathcal{K} \subseteq \mathcal{B}_X$ with respect to which all eigenfunctions are measurable. Bourgain's uniform Wiener–Wintner theorem [1] asserts that for all $f \in L^\infty(X, \mu)$ which are orthogonal to $\mathcal{K}_X$, there is a conull set $X' \subseteq X$ such that for all $x \in X'$,

$$\lim_{n\to\infty} \sup_{\xi\in\mathbb{R}} \left| \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) e^{2\pi i k\xi} \right| = 0.$$

This result will be very useful in the proof of the following generalization of Corollary 1.1.

Theorem 5. *Suppose that $f \in L_0^\infty(X, \mu)$, and $\beta \in \mathbb{R}$. Then there is a conull set $X' \subseteq X$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (\beta + f(T^k x))\, \delta_{\widetilde{M}_{k,n}(x)} = \beta\, m_{[0,1]},$$

in the weak topology on $\mathcal{M}(\mathbb{R})$.

Proof. The line of proof is very similar to the proof of Corollary 1.1, and is divided into two parts. We first assume that f is in the orthogonal complement of the Kronecker factor, and prove that for all $\phi \in \mathcal{S}$, with compactly supported Fourier transform $\hat{\phi}$, there is a conull set $X_\phi \subseteq X$ such that for all $x \in X_\phi$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k x)\, \phi(t_{k,n}) = 0.$$

To prove this, we note that,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k x)\phi(t_{k,n}) &= \lim_{n\to\infty} \int_{\mathbb{R}} \left[\frac{1}{n} \sum_{k=1}^{n} f(T^k x) e^{i\xi t_{k,n}} \right] \hat{\phi}(\xi)\, dm(\xi) \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} \left[\frac{1}{n} \sum_{k=1}^{n} f(T^k x) e^{i\xi k} \right] n\hat{\phi}(n\xi)\, dm(\xi).\end{aligned}$$

By Bourgain's uniform Wiener–Wintner theorem, the expression in the bracket converges uniformly to 0 in ξ as $n \to \infty$ for all $x \in X''$. Thus, by Hölder's inequality,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k x)\phi(t_{k,n}) = 0,$$

for all $x \in X''$, which proves the first claim.

Our second claim concerns functions f which are $\mathcal{K}$-measurable. Suppose that ψ is an eigenfunction of $(X, \mathcal{B}_X, \mu_X, T)$, with eigenvalue $\lambda \neq 1$, then, for all compactly supported $\phi \in \mathcal{S}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \psi(T^k x)\, \phi(t_{k,n}) = \lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} \lambda^k \phi(t_{k,n}) \right) \psi(x) = 0,$$

which is easy to check, and the details are left to the reader. Thus, for any finite linear combination of eigenfunctions f, there is a conull set $X' \subseteq X$ such that for all $x \in X'$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k x)\delta_{t_{k,n}} = 0,$$

in the weak topology on $\mathcal{M}(\mathbb{R})$. We will now boost this to hold for all essentially bounded which are measurable with respect to the Kronecker factor. Let f be essentially bounded and $\mathcal{K}$-measurable. By the definition of $\mathcal{K}$, there is a sequence of finite linear combinations of eigenfunctions, which we will denote by f_N, such that f_N converge to f almost everywhere on X and in L^1-norm. We claim that the random measures,

$$\nu_n^{x,f} = \frac{1}{n} \sum_{k=1}^{n} f(T^k x)\delta_{t_{k,n}} = 0,$$

converge to the zero measure on [0, 1] for all x in some conull set of X. Since we already know this convergence for the measures ν_n^{x,f_N} on some conull set $X' \subseteq X$ which we can take to be independent of N and containing the generic points for f, it suffices to prove that for all $x \in X'$,

$$\lim_{n\to\infty} |\nu_n^{x,f_N}(\phi) - \nu_n^{x,f}(\phi)| = 0,$$

for all compactly supported $\phi \in \mathcal{S}$. To prove this, note that, for all $x \in X'$,

$$\begin{aligned}\overline{\lim_{n\to\infty}} |\nu_n^{x,f_N}(\phi) - \nu_n^{x,f}(\phi)| &\le \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |f_N(T^k x) - f(T^k x)| \|\phi\|_\infty \\ &= \|\phi\|_\infty \int_X |f - f_N| \,\mathrm{d}\mu \to 0,\end{aligned}$$

which proves the last claim.

Finally, for a general $f \in L_0^\infty(X)$, we can decompose it into a direct sum of an essentially bounded $\mathcal{K}$-measurable function and an essentially bounded function orthogonal to $\mathcal{K}$. Since the theorem holds in both classes of functions, we are done. $\square$

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